An analytical solution to the inverse consumption function with liquidity constraints

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Abstract

This paper analyzes the consumption function induced by liquidity constraints under perfect foresight in continuous time, obtaining an analytical expression for the inverse consumption function. Liquidity constraints have dramatic effects on consumption, even long before the constraints bind.

Keywords: Liquidity constraints; Consumption; Optimal control

JEL classification: D91; E21; C61

1. Introduction

In general, there is no closed form solution for a liquidity constrained consumer problem even without uncertainty. Thus, researchers have relied on numerous solution algorithms to analyze this problem. Seater (1997) derives a limited optimal control solution to this problem, but shows only that consumption has a negative monotonic relationship with the ‘total’ costate variable.

In this paper, I present a closed form solution to the inverse consumption function using the constant relative risk aversion (CRRA) utility, which allows the effects of liquidity constraints on consumption to

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1 Mariger (1987) provides a geometric solution algorithm for a discrete time and finite horizon liquidity constrained problem.
be directly analyzed. I also show that the relatively high time preference rate causes consumption to fall over time and more importantly that the constraints cause initial consumption to fall relative to the case without the constraints. Moreover, I show that the rate of the reduction of consumption is the same as in the unconstrained case as long as the consumer has positive wealth.

2. The problem

A liquidity constrained consumer solves

\[ V(A_0) = \max_{\{C_t\}} \int_0^\infty e^{-\rho t} \frac{C_t^{1-s}}{1-s} \, dt, \quad (1) \]

subject to the usual budget constraint plus the endpoint conditions

\[ \dot{A}_t = rA_t + Y_t - C_t, \quad (2) \]

\[ A(0) = A_0, \quad (3) \]

\[ \lim_{t \to \infty} A_t = 0, \quad (4) \]

and also subject to the liquidity constraint

\[ A_t \geq 0 \quad \text{for all} \quad t \geq 0, \quad (5) \]

where \( C_t \) is consumption, \( Y_t \) is labor income, which is fixed and normalized at 1, \( A_t \) is wealth, \( s \) is the elasticity of intertemporal substitution, \( \rho \) is the rate of time preference, and \( r \) is the real interest rate. As in Deaton (1991) and Carroll (1997), consumers are assumed to be impatient enough to borrow or spend their assets to support current consumption. The impatience condition requires \( \rho > r \).

The current value Hamiltonian is

\[ H = \frac{C_t^{1-s}}{1-s} + \lambda_t [rA_t + 1 - C_t] + \mu_t A_t, \quad (6) \]

where \( \lambda \) and \( \mu \) are the costate variables for the budget constraint and the liquidity constraint, respectively. The necessary conditions in addition to the given constraints are

\[ H_c = C_t^{s} - \dot{\lambda}_t = 0, \quad \text{so} \quad C_t = \lambda_t^{-s}; \quad (7) \]

\[ \dot{\lambda}_t - \rho \lambda_t = -H_A = -r \lambda_t - \mu_t; \quad (8) \]

\[ \mu_t \geq 0, \quad \mu_t A_t = 0. \quad (9) \]
3. The solution

Kamien and Schwartz (1991, Part II, Section 17) provide two methods for solving optimal control problems with state variable inequality constraints. Seater (1997) derives a non-closed-form solution using the second of the two methods. This paper uses the first method to obtain a closed form solution to the inverse consumption function.

Suppose the liquidity constraint never binds, i.e., $A_t > 0$ for all $t$. Then from (9), $\mu_t = 0$ for all $t$. This implies from (8) that $\frac{d}{dt} \ln A_t = (\rho - r)$. Since (7) implies $\frac{d}{dt} \ln C_t = -s \frac{\lambda}{A_t}$,

$$\frac{\dot{C}_t}{C_t} = -s(\rho - r), \quad \text{so} \quad C_t = C_0 e^{-s(\rho - r)t}. \quad (10)$$

Plugging (10) into the lifetime budget constraint gives

$$A_0 + \frac{1}{r} = \int_0^\infty C_t e^{-rt} dt = C_0 \int_0^\infty e^{-[r+s(\rho - r)]t} dt = \frac{1}{r+s(\rho - r)} C_0. \quad (11)$$

Hence

$$C_0 = [r + s(\rho - r)] \left[ A_0 + \frac{1}{r} \right], \quad (12)$$

which is in fact the initial optimal consumption in the absence of the liquidity constraint. Using (2) and (12), the time path of wealth is found by

$$A_t = \left[ A_0 + \frac{1}{r} \right] e^{-s(\rho - r)t} - \frac{1}{r}. \quad (13)$$

Because of the impatience condition $\rho > r$, $A_t$ falls over time and must eventually drop below zero. This contradicts the assumption that $A_t \geq 0$ for all $t$. Therefore, the liquidity constraint must be active at some finite future time.

When wealth reaches zero, the consumer will spend at a rate equal to labor income due to the impatience condition. Thus, the optimal consumption plan after depleting wealth is to consume all labor income, while the optimal consumption plan before reaching zero wealth follows the consumption rule with no liquidity constraints, i.e., the optimal consumption plan is

$$C_t \begin{cases} C_0 e^{-s(\rho - r)t} & \text{if } t \in [0,T) \\ 1 & \text{if } t \in [T, \infty) \end{cases}, \quad (14)$$

where $T$ is the first moment for which $A_t = 0$. The continuity of consumption implied by the continuity of $\lambda$ up to the moment the liquidity constraint binds gives

$$C_T = 1 \iff C_0 e^{-s(\rho - r)T} = 1 \quad (15)$$

Hence

$$T = \frac{\ln C_0}{s(\rho - r)}. \quad (16)$$
Plugging (14) and (16) into the lifetime budget constraint gives

\[ A_0 + \frac{1}{r} = \int_0^\infty C_t e^{-rt} dt \]

\[ = C_0 \int_0^T e^{-[\rho+r(s(\rho-r))]t} dt + \int_T^\infty e^{-rt} dt \]

\[ = \frac{1}{s(\rho-r)+r} \left[ C_0 + \frac{s(\rho-r)}{r} \frac{e^{-[\rho+r(s(\rho-r))]T}}{C_0} \right], \tag{17} \]

which is the inverse of the initial optimal consumption.\(^2\) (14) and (17) imply that the initial optimal consumption is equal to or greater than the normalized labor income with equality when the initial wealth is zero, i.e., \( C_0 = 1 \) only when \( A_0 = 0 \).

Differentiating (17) with respect to \( C_0 \) and taking the reciprocal give a nonnegative marginal propensity to consume (MPC):

\[ \frac{dC_0}{dA_0} = [s(\rho-r)+r] \left[ 1 - C_0 \frac{s(\rho-r)+r}{C_0} \right]^{-1} \geq s(\rho-r)+r, \tag{18} \]

since \( C_0 \geq 1 \). The equality holds only when \( A_0 = \infty \), i.e., \( C_0 = \infty \) and \( \frac{dC_0}{dA_0} = \infty \) when \( A_0 = 0 \), i.e., \( C_0 = 1 \).\(^3\)

The second derivative for consumption with respect to wealth is

\[ \frac{d^2C_0}{dA_0^2} = -\frac{[s(\rho-r)+r]^3}{s(\rho-r)} \left[ 1 - C_0 \frac{s(\rho-r)+r}{C_0} \right]^3 \frac{s(\rho-r)+r}{C_0} \leq 0, \tag{19} \]

since \( C_0 \geq 1 \). The equality holds only when \( A_0 = \infty \) and \( \frac{d^2C_0}{dA_0^2} = -\infty \) when \( A_0 = 0 \). These two analytical results coincide with Carroll and Kimball (2001) that the consumption function under liquidity constraints is increasing and concave in wealth.

The exact shape of the consumption function is obtained based on the inverse solution (16), as shown by the solid line in Fig. 1, where the dashed line represents the unconstrained consumption function. This figure shows that the consumption function becomes concave when liquidity constraints are imposed. This result confirms Deaton (1991), in which the behavior induced by liquidity constraints is similar to that induced by a precautionary motive for saving. Moreover, the constrained consumption level approaches the unconstrained consumption level as wealth increases to infinity, making the constraint irrelevant. Seater (1997) argues that Deaton’s (1991) result of falling consumption under liquidity constraints results from the impatience condition \( \rho > r \), instead of the constraint itself. However, we need to distinguish between the initial drop in consumption and the declining consumption over time.

The solid line in Fig. 2 shows that optimal consumption under liquidity constraints is decreasing over time at the same rate as in the unconstrained case – shown by the dashed line – until the constraint is binding, i.e. wealth falls to zero. Once the constraint is binding, it exactly follows the income path.

\(^2\) This solution method is consistent with Mariger (1987) and easily carries over to the model with labor income growing constantly by normalizing all variables by the initial income. For this extension, see Park (2004).

\(^3\) This implies that a consumer will plan to spend the extra wealth infinitely fast when she is at zero wealth.
Fig. 1. Optimal consumption functions ($s=0.5$, $\rho=0.05$, $r=0.04$).

Fig. 2. Consumption paths ($s=0.5$, $\rho=0.05$, $r=0.04$, $A_0=1$).
Seater (1997) argues that the liquidity constraint actually causes consumption to fall less rapidly over time than it would without the constraint, but this can be true only if the constraint starts to bind. Although liquidity constraints exist, the rate of the reduction of consumption is the same as in the unconstrained case when the consumer has positive wealth. This conclusion coincides with Attanasio (1999).

4. Conclusion

This paper derives an analytical solution to the perfect foresight consumer model with liquidity constraints. The exact shape of the consumption function is obtained using an inverse solution and the results in the existing literature are proved analytically. This paper can serve as a “stepping-stone” for a better analytical understanding of the buffer-stock saving model by using insights from the corresponding certainty model. Furthermore, the solution may provide a good initial guess for complicated numerical solutions and simulations to the buffer-stock saving model.

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References